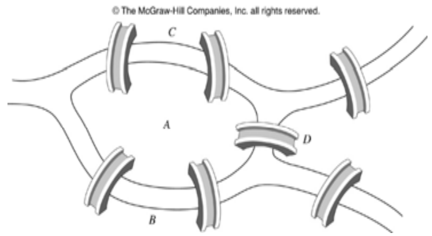
Euler and Hamilton Paths

Section 9.5

The Town of Konigsberg

- Is it possible to cross each bridge exactly once?



Solution by Euler

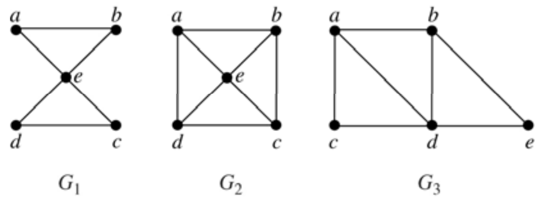
- Being the first to solve the problem
  - Might also be the first time that a graph theory is used
- Convert the problem into a multigraph
  - Find an Euler Path

Euler Path and Euler Circuit

- Definition:
  - An ***Euler circuit*** in a graph G is a simple circuit containing every edge of G.
  - An ***Euler path*** in G is a simple path containing every edge of G
- In short, it is a circuit/path containing all edges

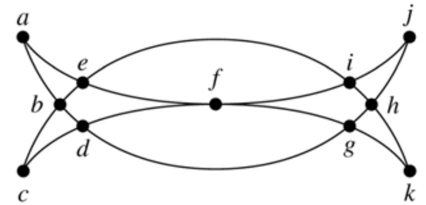
Example

- Which of these graphs have an Euler circuit/path?



Existence Condition

- A necessary and sufficient condition for a graph to have an Euler circuit
  - Each vertex must has even degree



Note: Condition

- What is "necessary" and "sufficient"??
- We say that *P* is a necessary condition for *Q* when *Q*→*P*
  - i.e., ~*P*→~*Q*
    - Hence the word "necessary"
- We say that *P* is a sufficient condition for *Q* when *P*→*Q*
  - i.e., if *P* is true, then *Q* must be true
- When P is a "necessary" and "sufficient"
  - *P*→*Q* and *Q*→*P*
  - i.e., *P* ⇔ *Q*

Theorem on Euler circuit

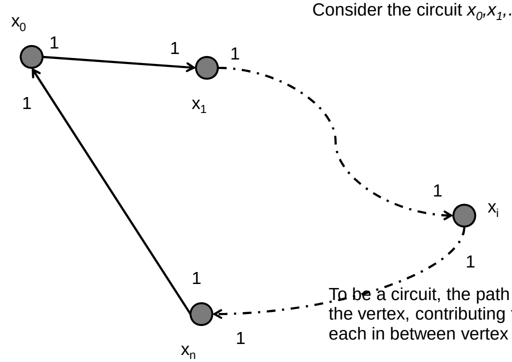
- A connected multigraph has an Euler circuit if and only if each of its vertices has even degree

Proof

- Since the theorem says “if and only if”
  - We need to proof both direction
  - i.e., we must show that
    - If a graph has an Euler circuit, every vertex must has even degree
      - The necessary side
    - If every vertex has even degree, the graph must has an Euler circuit.
      - The sufficient side

Proof for necessity condition

Consider the circuit $x_0,x_1,\ldots,x_i,\ldots x_n,x_0$



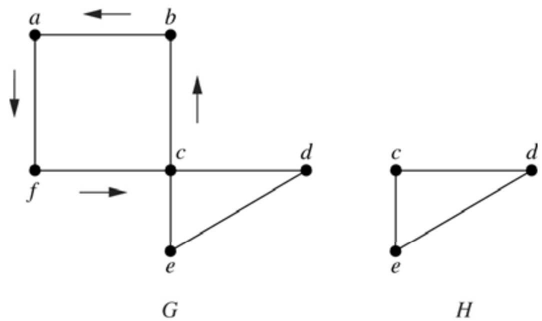
To be a circuit, the path must go through the vertex, contributing two degree for each in between vertex

since it is a circuit, the first and the last vertex is the same, thus the degree of the first vertex must also be even

Proof for sufficient

- Proof by construction
- If all vertices has even degree
  - We could find a simple circuit (any circuit would do)
    - This is possible because we use one degree to enter a vertex and one additional degree to exit the vertex
  - If the circuit contains all edges, the proof is complete
  - If that is not the case, we take out every edge in the circuit and any vertex having no edge
    - The remaining graph must have at least one vertex in common
    - We repeat the same step on the remaining graph
    - Use the common vertex to join the circuit

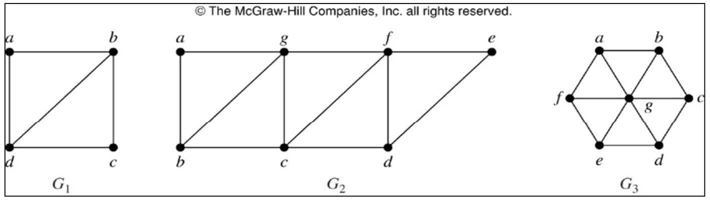
Proof for sufficient condition



Theorem on Euler path

- A connected multigraph has an Euler path if and only if it has exactly two vertices of odd degree
  - We need two vertices for start and stop

Example

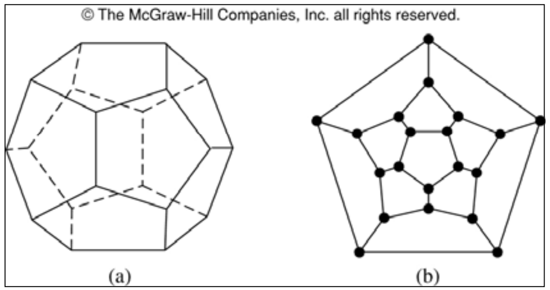


Hamilton Path

- Definition
  - A path $x_0, x_1, \ldots, x_n$ in the graph $G = (V,E)$ is called a ***Hamilton path*** if $V = \{x_0, x_1, \ldots, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j < n$.
  - A circuit $x_0, x_1, \ldots, x_n, x_0$ (with $n > 1$)in a graph $G = (V,E)$ is called a ***Hamilton circuit*** if is a Hamilton path
- In short, it is a path containing all vertices

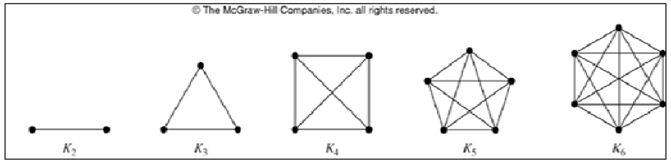
Origin

- A puzzle by a mathematician William Rowan Hamilton "A Voyage Round the World"



Example

- K_n



Existence Condition

- Sadly, we don't have that useful "necessary and sufficient condition" for Hamilton path
- At least, we have some condition

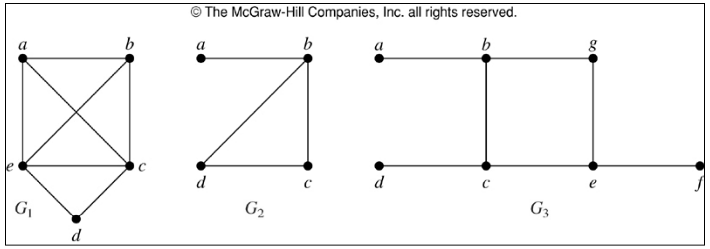
Dirac's Theorem

- If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit

Ore's Theorem

- If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

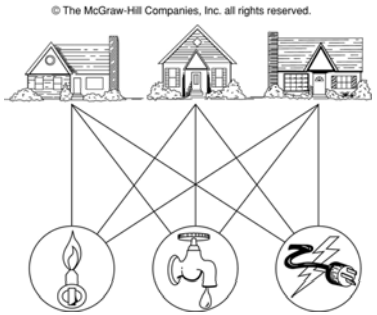
Example



Planar Graphs

Section 9.7

The utility routing problem

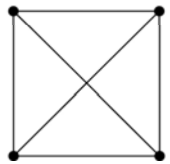


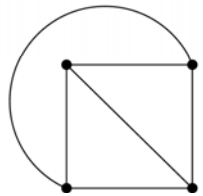
- i.e., can K3,3 be drawn in the plane so that no two of its edges cross

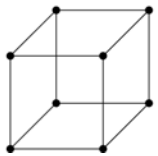
Planar Graph

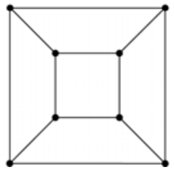
- Definition
  - A graph is called ***planar*** if it can be drawn in the plane without any edges crossing
    - where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.
    - Such a drawing is called a planar representation of the graph

Example of Planar Graph



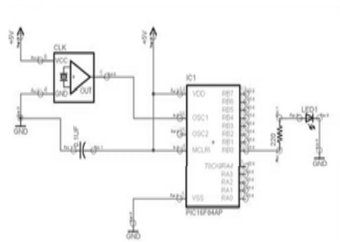


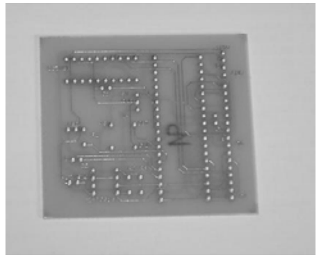




Application

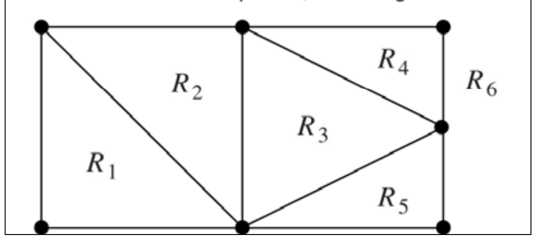
- Circuit Design





Region

- Region is the area surrounded by edges
  - Also include the outmost "unbound" area



Eule's Formular for Planar Graphs

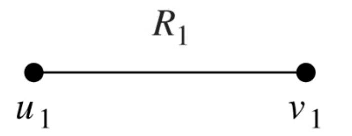
- Let *G* be a connected planar simple graph with ***e*** edges and ***v*** vertices. Let ***r*** be the number of regions in a planar representation of *G*.
  - Then ***r = e − v + 2***

Proof

- By Induction
- Start from G
  - Take out one edge (also take out any floating vertex)
    - We get Gn
  - Take out another edge (also take out any floating vertex)
    - We get G(n-1)
  - And so on… until G_0
  - We get a sequence G_0,G_1,…,G_n = G
- Will show that
  - (a) Euler formula is correct for G_0
  - (b) If G_k is correct, then G_{k+1} is correct as well

Proof

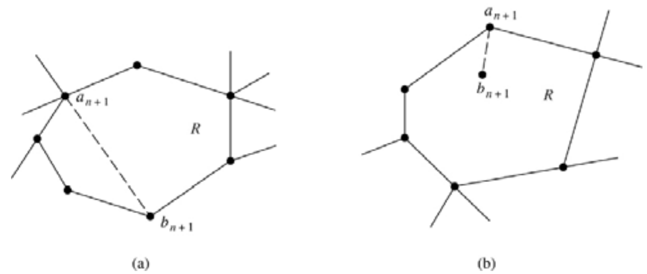
- The base case



Proof

- From G_k to G_{k+1}
  - G_{k+1} can be constructed from G_k by adding the removed edge
  - Two cases
    - (a) The new edge connects two vertices in G_k
    - (b) The new edge connects one vertex in G_k and a new vertex
  - The formula is correct for both cases

Proof



Corollary 1

- If G is a connected planar simple graph with ***e*** edges and ***v*** vertices where ***v ≥ 3***,
  - then ***e ≤ 3v - 6***

Corollary 2

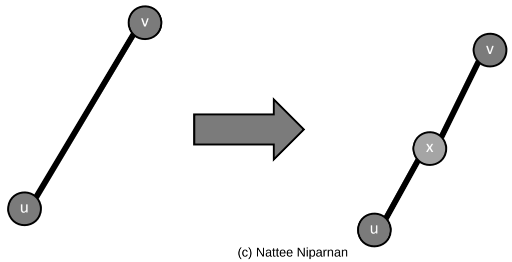
- If G is a connected planar simple graph, then G has a vertex of degree not exceeding five

Corollary 3

- If G is a connected planar simple graph has ***e*** edges and ***v*** vertices with ***v ≥ 3*** and no circuits of length three, then ***e ≤ 2v - 4***

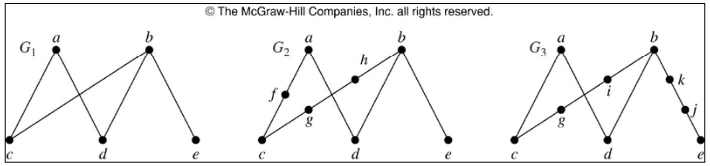
Elementary Subdivision

- An operation on a graph
  - Adding a vertex on the edge
    - Formally, remove an edge *(u,v)*
    - add edge *(u,x)* and *(x,v)* and add vertex *x*



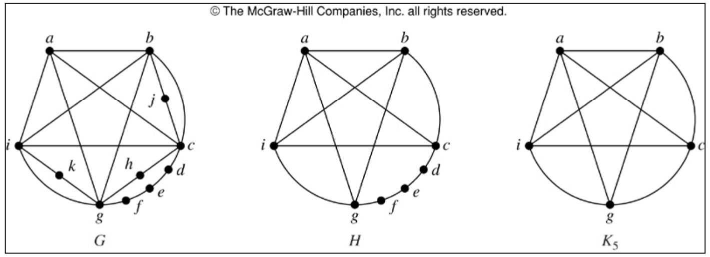
Homeomorphic

- G1 and G2 are Homeomorphic if we can obtain one from the other using only elementary subdivision

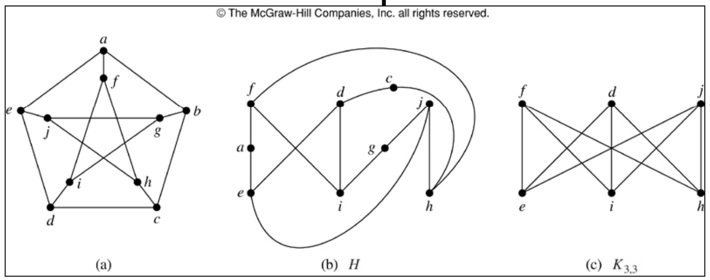


Kuratowski's Theorem

- A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5



Example

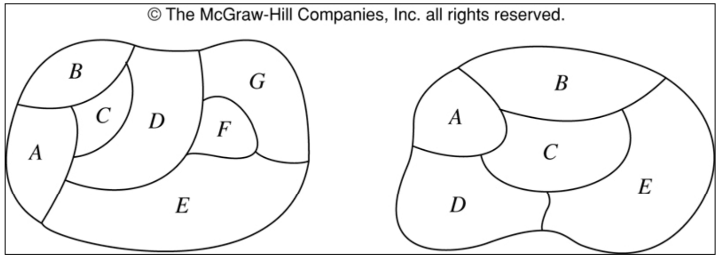


- Take b out
  - Result in H
- Use elementary subdivition to remove a,c and g

Graph Coloring

Section 9.8

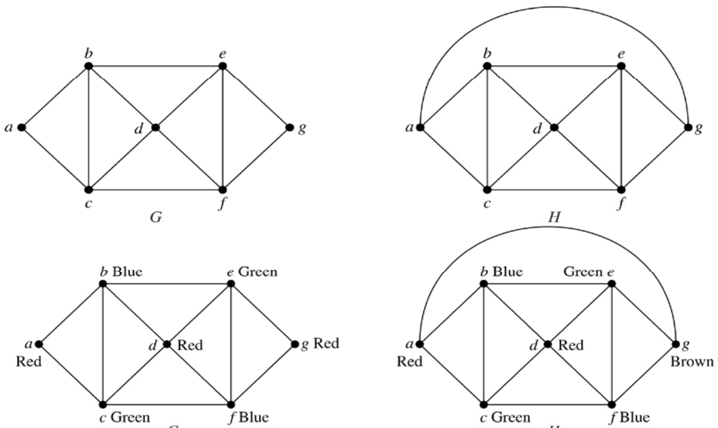
Map Coloring Problem



Graph Coloring

- Definition
  - A **coloring** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color
  - The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph

Example



The Four Color Theorem

- The chromatic number of a planar graph is no greater than four

Example

- What is the chromatic number of K_n?

Application

- Final Exam Scheduling