International School of Engineering
Faculty of Engineering Chulalongkorn University
2143101 Discrete Mathematics
Second Semester, Final Examination, May 6th 2009 Time 13.00 – 16.00

Name……………………………………………………………Student ID…………………………… No. in CR…………..

Instruction

1. There are 3 parts with the total of 22 questions in this exam paper. The exam has 13 pages.
2. **Answer of each part must be written on the corresponding pages only (you can use the back of the page)**
   a. **Part 1: page 3 – 5**
   b. **Part 2: page 7 – 9**
   c. **Part 3: page 11 – 13**
3. **Write your student ID, full name, and your number in CR58 in the space provided on the top of every page or on the first page of given exam booklets.**
4. No documents and no calculators are allowed inside the exam room.
5. No borrowing is allowed unless it is supervised by the proctor.
6. You must not bring any part of this exam paper outside. The exam paper is a government's property. Violators will be prosecuted under a criminal court.
7. Student who wishes to leave the exam room before the end of the exam period must raise their hand and ask for permission before leaving the room. Student must leave the room in the orderly manner.
8. Once the time is expired, student must stop writing and must remain seated quietly until the proctors collect all the exam papers or given exam booklets. Only then, the students will be allowed to leave the room in the orderly manner.
9. Any student who does not obey the regulations listed above will receive punishment under the Faculty of Engineering Official Announcement on January 6, 2009 regarding the exam regulations.
   a. **With implicit evidence or showing intention for cheating, student will receive an F in that subject and will receive an academic suspension for 1 semester.**
   b. **With explicit evidence for cheating, student will receive an F in that subject and will receive an academic suspension for 1 year.**

I understand and agree to the given instructions

Signature (………………………………………………..)

Part 1

1. (1 point) How many divisions are required to find gcd(212,34) using the Euclidean algorithm?
2. (1 point) Which integers are divisible by 5 but leave a remainder of 1 when divided by 3?
3. (1 point) How many zeroes are there at the end of 100! ?
4. (2 points) Prove that if n is an odd positive integer, then $n^2 \equiv 1\ (\mathbf{mod}\ 8)$
5. (2 points) Show that if $2^n - 1$ is prime, then n is prime. [Hint: Use the identity $2^{ab} - 1 = (2^a - 1) \cdot (2^{1(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1)$]
6. (2 points) Use Algorithm 5 (Modular Exponentiation) to find $7^{644}\ \mathbf{mod}\ 645$
7. (2 points) Find the solutions to the system $x \equiv 1\ (\mathbf{mod}\ 4), x \equiv 2\ (\mathbf{mod}\ 5), x \equiv 3\ (\mathbf{mod}\ 7)$
8. (3 points) Show that if m is a positive integer greater than 1 and $ac \equiv bc (\mathbf{mod}\ m)$, then $a \equiv b\ \mathbf{mod}\ m/\text{gcd}\ (c, m)$

Part 2

9. (1 point) A simple graph is called **regular** if every vertex of this graph has the same degree. For which values of m and n is $K_{m,n}$ regular?
10. (1 point) How many edges does a full binary tree with 1,000 internal vertices have?
11. (1 point) Suppose that the address of the vertex v in the ordered rooted tree T is 3.4.5.2.4 (using the **universal addressing system**). What is the smallest possible number of vertices in T?
12. (2 points) Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that
    a) $2e/v \geq m$
    b) $2e/v \leq M$
13. (2 points) The **converse** of a directed graph $G = (V, E)$ is the directed graph (V, F) where $(u, v) \in F$ if and only if $(v, e) \in E$. Show that if G and H are isomorphic directed graphs, then the converses of G and H are also isomorphic.
14. (2 points) Determine whether the graph in Fig. 1 is homeomorphic to $K_{3,3}$.
15. (2 points) The binomial tress $B_i, i = 1,2,3, \ldots$, are ordered rooted trees defined recursively:

    Basis step: The binomial tree B_0 is the tree with just a single vertex

    Recursive step: Let k be a nonnegative integer. To construct the binomial treeB_{k+1}, add a copy of B_k to a second copy of B_k by adding an edge that makes the root of the first copy of B_k be the leftmost child of the second copy of B_k.

    Draw B_k for $k = 0,1,2,3,4$.
16. (3 points) An orientation of an undirected simple graph is an assignment of directions to its edges such that the resulting directed graph is strongly connected. When an orientation of an undirected graph exists, this graph is called orientable. Show that a graph is not orientable if it has a cut edge.

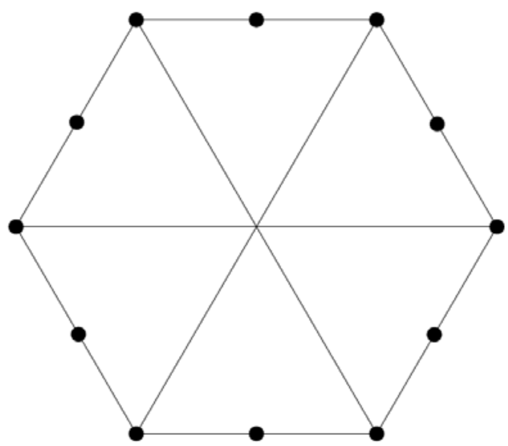

Figure 1: Graph for problem 14.

Part 3

17. (2 points) Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

which relates terms in Pascal's triangle that form a hexagon.

18. (2 points) How many ways are there to seat six boys and eight girls in a row of chairs so that no two boys are seated next to each other?
19. (2 points) Find the explicit formula for the Fibonacci numbers where $n \geq 3$ and $F_0 = F_1 = 1$.
20. (2 points) Prove that the derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

21. (3 points) Let (x_i,y_i), i = 1,2,3,4,5, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinate.
22. (3 points) Find all solutions of the recurrence relation $a_n = 7a_{n-1}-16a_{n-2}+12a_{n-3}+n4^n$, with the initial condition $a_0=-2$, $a_1=0$ and $a_2=5$